ISRAEL JOURNAL OF MATHEMATICS **93** (1996), 189–194

NILPOTENT SUBGROUPS OF SELF EQUIVALENCES OF TORSION SPACES

BY

KAI XU

Department of Mathematics, National University of Singapore, Singapore 0511
e-mail: matxukai@leonis.nus.sg

ABSTRACT

Let X be a finite p-torsion based connected nilpotent CW-complex. We give a criterion of a subgroup of $\mathcal{E}(X)$, the group of self equivalences of X, to be a nilpotent group, in terms of its action on $E_*(X)$, where E is a CW-spectrum, satisfying some technical conditions.

Recall that a spectrum E is **connective** if there is an integer k such that $\pi_i(E) = 0$ if $i < k$ and $\pi_k(E) \neq 0$. We write $ct(E) = k$.

Consider the induced map from $\mathcal{E}(X)$, the group of based self equivalences of a space X, to $\prod_{i=-\infty}^{+\infty} \mathrm{Aut}(E_i(X))$. Each subgroup in $\mathcal{E}(X)$ acts on $E_*(X)$ in the natural way.

Let X be a based connected finite nilpotent CW-complex, its integral homology theory is of p-torsion and $\pi_k(E) \otimes \mathbb{Z}_{(p)} \neq 0$, where p is a fixed prime and E be a connective CW-spectrum with $ct(E) = k$. The main purpose of this note is to prove the following theorem.

THEOREM 0.1: *If a subgroup $G \subseteq \mathcal{E}(X)$ acts on $E_i(X)$ nilpotently for all i, then G is nilpotent.*

This result can be thought of a generalisation of Theorem 3.5 of [4]. Namely, the assumption of E being a ring spectrum in it has been removed.

Received August 2, 1994

1. A reduction of Theorem 0.1

Let G be a subgroup of $\mathcal{E}(X)$, satisfying the hypothesis in Theorem 0.1. For $j \geq 0$, write α_j for the induced map:

$$[\Sigma^j X, \Sigma^j X] \longrightarrow \prod_{i>0} \operatorname{End} H_i(\Sigma^j X, \mathbb{F}_p),$$

where j is allowed to be zero by indentifying $\Sigma^0 X$ as X. Denote by $O_p(C)$ the Fitting p-subgroup of a group C. Recall a result in [1]:

$$f \in O_p(\mathcal{E}(X)) \quad \text{if and only if } \alpha_0(f) \in O_p\{\alpha_0(\mathcal{E}(X))\}. \tag{1}$$

Then we have

$$\frac{G}{G \cap O_p(\mathcal{E}(X))} \cong \frac{\alpha_0(G)}{\alpha_0(G) \cap O_p\{\alpha_0(\mathcal{E}(X))\}}.$$

Since Fitting p-subgroups are nilpotent, to show G is a nilpotent group, it suffices to prove $\alpha_0(G)$ is.

For simplicity, we denote by Σ^s both the s-fold suspension map and its induced map in homology. Write $\Sigma^s(C)$ for the subgroup consisting of the image elements of Σ^s on a group C. Then Σ^s: $\alpha_0(G) \longrightarrow \Sigma^s\alpha_0(G)$ being injective and the relation: $\Sigma^s\alpha_0(G) = \alpha_s\Sigma^s(G)$ imply that $\alpha_0(G)$ is isomorphic to a subgroup of $\alpha_s\Sigma^s(G)$. So we only need prove the last group is nilpotent. In the following we let s be sufficiently large, so that we can work in the stable category.

2. Nilpotent groups in stable case

Recall a finite p-torsion space X is atomic if each self map of it is either nilpotent or an equivalence. By [7], [8], we have a splitting:

$$\Sigma^j X \simeq Y_1 \bigvee \cdots \bigvee Y_n, \tag{2}$$

where $j \geq 1$ and Y_i is a wedge of atomic spaces of the same connectivities and the connectivity of Y_i, say, k_i, is strictly increasing with i. Denote by λ_j the induced map:

$$[\Sigma^j X, \Sigma^j X] \longrightarrow \prod_{i=1}^{n} \operatorname{End} H_{k_i}(\Sigma^j X, \mathbb{F}_p).$$

LEMMA 2.1: *If $t \in \mathcal{E}(\Sigma^s X)$ and s is sufficiently large, then $\lambda_s(t)$ is nilpotent $\iff$ $\alpha_s(t)$ is.*

Proof: The direction '$\Longleftarrow$' is clear. Now we show the opposite direction. Assume $\lambda_s(t)$ is nilpotent. Since $[\Sigma^s X, \Sigma^s X]$ is finite, there is an integer m such that $t^{2m} = t^m$. We wish to show that t^m is the constant map. For each element $g \in [\Sigma^s X, \Sigma^s X]$, we define the mapping telescope:

$$(\Sigma^s X)_g := \coprod(\{r\} \times I \times \Sigma^s X)/\sim$$

where $I = [0,1]$, the disjoint union is taken over all non-negative integers r, and '$\sim$' represents the relations that $(r,1,y) \sim (r+1,0,g(y))$ and $(r,t,*) \sim *$ with the quotient topology, where $y \in \Sigma^s X$. Setting $g = t^m$ and $1 - t^m$, respectively, we obtain the mapping telescopes Z_1 and Z_2, respectively, and therefore a splitting: $\Sigma^s X \simeq Z_1 \bigvee Z_2$ such that

$$t^m\colon Z_1 \bigvee Z_2 \longrightarrow Z_1 \bigvee Z_2$$

is in the form $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Since the splitting (2) is unique up to the ordering [7], [8], either Z_1 is a point or the connectivity of Z_1 is k_i for some i with $1 \le i \le n$ (see (2)). On the other hand, that $\lambda(t^m) = 0$ implies $H_{k_i}(b, \mathbb{Z}) = 0$, where b is the following composition map,

$$Z_1 \xrightarrow{\text{incl.}} Z_1 \bigvee Z_2 \xrightarrow{t^m} Z_1 \bigvee Z_2 \xrightarrow{\text{proj.}} Z_1,$$

which is the identity. So Z_1 is a point and therefore t^m is constant. ∎

We write $HO_p(G)$ for the product group of H and $O_p(G)$. From [1], we know each element $f \in \alpha_s(\Sigma^s G)$ can be written as gh, where h is some element in $O_p\{\alpha_s(\mathcal{E}(\Sigma^s X))\}$ and $g \in \alpha_s(\mathcal{E}(\Sigma^s X))$ is block-diagonal: it sends $H_l(Y_i, \mathbb{F}_p)$ to itself for all given l and i (see (2)). All such elements g form a group Φ, say.

Then

$$\alpha_s(\Sigma^s G) O_p\{\alpha_s(\mathcal{E}(\Sigma^s X))\} \cong \Phi O_p\{\alpha_s(\mathcal{E}(\Sigma^s X))\}. \tag{3}$$

Recall that there is an Atiyah–Hizerbruch spectral sequence

$$E^2_{b,t} = H_b(Y_i, \pi_t(E)) \Longrightarrow E_{b+t}(Y_i), \tag{4}$$

where $i = 1, \ldots, n$. From the definitions of k_i and $ct(E)$, we obtain $E^2_{k_i,ct(E)} = E^\infty_{k_i,ct(E)}$. So that Φ acts nilpotently on $E_{k_i+ct(E)}(Y_i)$ implies it also acts nilpotently on $H_{k_i}(Y_i, \pi_{ct(E)}(E))$ for all i. Before carrying on, we first prove the following,

LEMMA 2.2: *Let A be an abelian group satisfying $A \otimes \mathbb{Z}_{(p)} \neq 0$. Then Φ acts nilpotently on $H_{k_i}(Y_i, A)$ if and only if it does on $H_{k_i}(Y_i, \mathbb{Z})$.*

Proof: The connectivity hypothesis: $H_{k_i-1}(Y_i, \mathbb{Z}) = 0$ implies

$$\operatorname{Tor}(H_{k_i-1}(Y_i, \mathbb{Z}), A) = 0.$$

So the Universal Coefficient Theorem gives rise to a natural isomorphism:

$$H_{k_i}(X, \mathbb{Z}) \otimes A \xrightarrow{\cong} H_{k_i}(X, A).$$

Hence the result follows from the assumption on A and that $H_{k_i}(X, \mathbb{Z})$ is a finite abelian p-group. ∎

COROLLARY 2.3: *Φ acts nilpotently on $H_{k_i}(Y_i, A) \Longleftrightarrow$ it does on $H_{k_i}(Y_i, \mathbb{F}_p)$.*

By setting $A = \pi_{ct(E)}(E)$ in the preceding corollary, we conclude Φ acts nilpotently on $H_{k_i}(Y_i, \mathbb{F}_p)$. We wish to show that Φ acts nilpotently on $H_l(\Sigma^s X, \mathbb{F}_p)$ for all l. Since it is block-diagonal, it is sufficient to prove that Φ acts nilpotently on $H_l(Y_i, \mathbb{F}_p)$ for all l and i.

To simplify the notations, we denote by $g_{l\sigma}$ the induced map $H_l(f_\sigma, \mathbb{F}_p)$ for a given element $f_\sigma \in [Y_i, Y_i]$. Let

$$S(m,l) = \{(g_{l1} - 1) \cdots (g_{lm} - 1) \mid g_{l\sigma} \in \Phi, \ \sigma = 1, \ldots, m\}$$

and $K(m,l)$ be the right ideal, generated by $S(m,l)$ in the finite ring R_s, the image of α_s.

Since Φ acts nilpotently on $H_{k_i}(Y_i, \mathbb{F}_p)$, by the definition of nilpotent action, we have, for a sufficiently large integer N,

$$K(N, k_i) H_{k_i}(Y_i, \mathbb{F}_p) = 0.$$

For a given element $t \in [Y_i, Y_i]$, we write $H_*(t, \mathbb{F}_p)$ for the induced map of t on homology. Lemma 2.1 tells us that $H_{k_i}(t, \mathbb{F}_p) - 1$ is nilpotent for all i if and only if $H_l(t, \mathbb{F}_p) - 1$ is for all l. Thus each element in $K(N, l)$ is nilpotent for all l. Since R_s is a finite therefore an Artinian ring, there is an integer q, such that $K(Nq,l) \subseteq K(N,l)^q = 0$, see, for example, the proposition of §4.4 in [5]. It follows that Φ acts nilpotently on $H_l(Y_i, \mathbb{F}_p)$ for all l by the definition of nilpotent action.

PROPOSITION 2.4: *Φ is a nilpotent group.*

Proof: Note that Φ is a subgroup of $\prod_{l>0} \operatorname{Aut} H_l(\Sigma^s X, \mathbb{F}_p)$ and it acts nilpotently on $\prod_{l>0} H_l(\Sigma^s X, \mathbb{F}_p)$. So it is nilpotent, see the discussion after Theorem A in [2], and §7.1.1.1 in [6]. ∎

To prove $\alpha_s(\Sigma^s G)$ is nilpotent, we have to show

$$\frac{\alpha_s(\Sigma^s G)}{\alpha(\Sigma^s G) \cap O_p\{\alpha_s(\mathcal{E}(X))\}}$$

is nilpotent. This follows from Proposition 2.4 and the following isomorphisms,

$$\begin{aligned}\frac{\alpha_s(\Sigma^s G)}{\alpha_s(\Sigma^s G) \cap O_p\{\alpha_s(\mathcal{E}(X))\}} &\cong \frac{\alpha_s(\Sigma^s G) O_p\{\alpha_s(\mathcal{E}(X))\}}{O_p\{\alpha_s(\mathcal{E}(X))\}} \\ &\cong \frac{\Phi O_p\{\alpha_s(\mathcal{E}(X))\}}{O_p\{\alpha_s(\mathcal{E}(X))\}} \cong \frac{\Phi}{\Phi \cap O_p\{\alpha_s(\mathcal{E}(X))\}},\end{aligned}$$

where the first and the last isomorphisms are implied by the Second Group Isomorphism Theorem and the second is obtained from (3). ∎

References

[1] M. C. Crabb, J. R. Hubbuck and K. Xu, *Fields of spaces*, in *Adams Memorial Symposium on Algebraic Topology*, Lecture Notes Series **175**, London Mathematical Society, Cambridge University Press, Cambridge, England, 1992, pp. 241–254.

[2] E. Dror and A. Zabrodsky, *Unipotency and nilpotency in homotopy equivalences*, Topology **18** (1979), 257–281.

[3] J. R. Hubbuck, *Self maps of H-spaces*, in *Advances in Homotopy Theory*, Lecture Notes Series **139**, London Mathematical Society, Cambridge University Press, Cambridge, England, 1989, pp. 105–110.

[4] K. I. Maruyama and M. Mimura, *Nilpotent subgroups of the group of self-homotopy equivalences*, Israel Journal of Mathematics **72** (1990), 313–319.

[5] R. S. Pierce, *Associative algebras*, in *Algebraic Systems*, GTM **88**, Springer-Verlag, New York, 1982.

[6] B. I. Plotkin, *Groups of Automorphisms of Algebraic Systems,* Wolters-Noordhoff Publishing, Groningen, The Netherlands, 1972.

[7] C. W. Wilkerson, *Genus and concellation*, Topology **14** (1975), 29–36.

[8] K. Xu, *Endomorphisms of complete spaces,* Thesis, Aberdeen University, 1991.